Steinberg Modules & Arithmetic Groups

Examples : $SL_n\mathbb{Z}$, $\Gamma_n(p) = \ker(SL_n\mathbb{Z} \xrightarrow{\text{mod } p} SL_n\mathbb{F}_p)$

$SL_n R$ R: no. ring (eg: $\mathbb{Z}[i]$, $\mathbb{Z}[\omega]$)

$Sp_{2n} R$ $\Gamma^{\omega}_{2n}(p) = \ker(Sp_{2n}R \longrightarrow Sp_{2n}(R/p))$

Convention: $\mathbb{Q}$-coeffs for H^*

The cohomology of these groups arises in topology, number theory, algebraic geometry, K-theory...

Goal of Talk: Give a flavour of algebraic & combinatorial tools that can be used to compute these H^*-groups

$SL_n R$:

$R=\mathbb{Z}$	$\text{rk } H^{\binom{n}{2}}(\Gamma_n(3)) = 3^{\binom{n}{2}}$ (Lee-Szczarba)	$\text{rk } H^{\binom{n}{2}}(\Gamma_n(5))$: recursive in n... $> 2^{n-1}\, 5^{\binom{n}{2}}$ (Miller-Patzt-Putman)
$R=\mathbb{Z}[i]$	$\text{rk } H^{n^2-n}(\Gamma_n(1+2i)) = 5^{\binom{n}{2}}$	[P.] $\text{rk } H^{n^2-n}(\Gamma_n(3))$: recursive in n... $> 2^{n-1}\, 9^{\binom{n}{2}}$
$R=\mathbb{Z}[\omega]$	$\text{rk } H^{n^2-n}(\Gamma_n(1+3\omega)) = 7^{\binom{n}{2}}$	[P.] $\text{rk } H^{n^2-n}(\Gamma_n(1+4\omega))$: recursive in n... $> 2^{n-1}(13)^{\binom{n}{2}}$

$Sp_{2n} R$:

$R=\mathbb{Z}$	[P.] $\text{rk } H^{n^2}(\Gamma^{\omega}_n(3)) = 3^{n^2}$
$R=\mathbb{Z}[i]$	[P.] $\text{rk } H^{2n^2-n}(\Gamma^{\omega}_n(1+2i)) = 5^{n^2}$
$R=\mathbb{Z}[\omega]$	[P.] $\text{rk } H^{2n^2-n}(\Gamma^{\omega}_n(1+3\omega)) = 7^{n^2}$

Q : What's special about these H^* degrees?
What is governing these calculations?
(Teaser: Depends on units of $R/(p)$ vs R ...)

Borel - Serre Duality

R : no. ring, $\Gamma <_{\text{fin ind.}} SL_n R$

$$H^{\nu - i}(\Gamma) \cong H_i(\Gamma ; \mathcal{D})$$

$\uparrow$ dualising module

$\nu = \nu(R)$: quadratic in n] "top degree"

$R = \mathbb{Z}$: $\nu = \binom{n}{2}$

$R = \mathbb{Z}[i]$ or $\mathbb{Z}[\omega]$: $\nu = n^2 - n$

This talk : R Euclidean domain, $\Gamma = \Gamma_n(p)$ $p \in R$ prime

$$H^{\nu}(\Gamma_n(p)) \cong H_0(\Gamma_n(p) ; \mathcal{D}) \cong \underline{\underline{(\mathcal{D})_{\Gamma_n(p)}}}$$

$\mathcal{D}$: The Steinberg Module

Steinberg Modules

representation of spl. linear group

$St_n R$ R : PID

$St_n R = \tilde{H}_{n-2}(T_n R)$

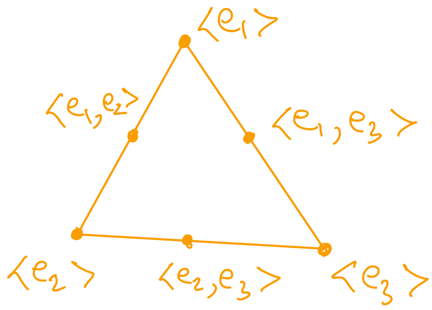
$\uparrow$ Tits building

Vertices of $T_n R$: $0 \subsetneq W \subsetneq R^n$ summands

k - simplices : $0 \subsetneq W_0 \subsetneq \cdots \subsetneq W_k \subsetneq R^n$

Note : $SL_n R \curvearrowright T_n R$

$\underline{n=3}$: Subcplex of $T_3 R$



apartment

$T_3 R$ built by "gluing such spheres together"

We are interested in $\boxed{(St_n R)_{\Gamma_n(p)}}$

$\underline{Q}$: How do we get bounds on $(St_n R)_{\Gamma_n(p)}$?

Fix $R = \mathbb{Z}$, $p \in \mathbb{Z}$

$T_n \mathbb{Z} \longrightarrow T_n \mathbb{F}_p$

$0 \subsetneq W_0 \subsetneq \cdots \subsetneq W_k \subsetneq \mathbb{Z}^n \xmapsto{\text{mod } p}$ flag in $\mathbb{F}_p^n$

$$\boxed{\rightsquigarrow \text{Get } (St_n \mathbb{Z})_{\Gamma_n(p)} \longrightarrow St_n \mathbb{F}_p : \text{rk } p^{\binom{n}{2}}}$$

$\underline{\text{Lee-Szczarba}}$: inj. when $p = 3$.

$\underline{\text{Input}}$: • Constructed a presentation (really, a resolution) of $St_n \mathbb{F}_p$.

• Inj for $p=3$ $\longleftrightarrow$ $\mathbb{Z}^{\times} \twoheadrightarrow \mathbb{F}_3^{\times}$

The Symplectic Group

$Sp_{2n}R$ has a similar duality story to SL_nR.

$R^{2n} = \langle e_1, f_1, \dots, e_n, f_n \rangle$

$\omega(e_i, f_j) = \delta_{ij} = -\omega(f_j, e_i) \qquad \omega(e_i, e_j) = 0 = \omega(f_i, f_j)$

$Sp_{2n}R$: form-preserving automorphisms.

$\Gamma^{\omega}_{2n}(p) \subset Sp_{2n}R = \ker\left(Sp_{2n}R \xrightarrow{\text{mod } p} Sp_{2n}(R/p)\right)$

Dualising module : $St^{\omega}_{2n}R$ "sympl. Steinberg Module"

$$St^{\omega}_{2n}R = \tilde{H}_{n-1}(T^{\omega}_{2n}R)$$

k-simplices $\leftrightarrow 0 \subsetneq W_0 \subsetneq \cdots \subsetneq W_k \subset R^{2n}$

isotropic ($\omega|_W \equiv 0$) summands

Want to study $\boxed{(St^{\omega}_{2n}R)_{\Gamma^{\omega}_{2n}(p)}}$

Fix $R = \mathbb{Z}$.

Have a surjection $\boxed{(St^{\omega}_{2n}\mathbb{Z})_{\Gamma^{\omega}_{2n}(p)} \twoheadrightarrow St^{\omega}_{2n}(\mathbb{F}_p) : \text{rk } p^{n^2}}$

Need a presentation of $St^{\omega}_{2n}(\mathbb{F}_p)$ to study injectivity.

Lee-Szczarba's SL-construction fails...

$\rightsquigarrow$ <u>Genus filtration</u> : Note $\underbrace{T^{\omega}_{2n}(\mathbb{F})}_{\substack{\text{spanned by } W \\ \text{s.t. } \omega|_W \equiv 0}} \subset T_{2n}(\mathbb{F})$

For $W \subset \mathbb{F}^{2n}$, let $\boxed{\text{genus of } W := \frac{1}{2} rk(\omega|_W)}$

Filter $T_{2n}(\mathbb{F})$ by genus and take s.s.

$$0 \to St_{2n}\mathbb{F} \to \cdots \to \bigoplus_{\substack{g(W)=2 \\ W \text{ sympl.}}} St(W) \otimes St^{\omega}(W^{\perp}) \longrightarrow \bigoplus_{\substack{g(W)=1 \\ W \text{ sympl.}}} St(W) \otimes St^{\omega}(W^{\perp}) \longrightarrow St^{\omega}_{2n}\mathbb{F} \longrightarrow 0$$

$\to$ inductively get resolution of $St^{\omega}_{2n}\mathbb{F}$

$\to$ Use it to show $H^{n^2}(\Gamma^{\omega}_{2n}(3))$

$$\boxed{\cong (St^{\omega}_{2n}\mathbb{Z})_{\Gamma^{\omega}_{2n}(3)} \cong St^{\omega}_{2n}\mathbb{F}_3}$$

$\boxed{Q}$ What does $T_n\mathbb{Z} \longrightarrow T_n\mathbb{F}_p$ forget?

<u>Eg</u>: $n=2$, $p=5$

$$\underbrace{\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle, \left\langle \begin{bmatrix} 2 \\ 5 \end{bmatrix} \right\rangle}_{T_2\mathbb{Z}} \longmapsto \underbrace{\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle = \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\rangle}_{T_2\mathbb{F}_5}$$

But $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ lie in different $\Gamma_2(5)$-orbits.

Want version of $T_2\mathbb{F}_5$ that distinguishes b/w $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$

Idea : Identify line generators upto ($\pm$)-sign

New complex : $T_n^{\pm}\mathbb{F}_p$ Vertices : (W, Ω)

$\Omega \in \Lambda^{top} V$, upto $\pm$ sign

Rmk : $T_n^{\pm}\mathbb{F}_3 = T_n\mathbb{F}_3$

$$(T_n\mathbb{Z})_{\Gamma_n(p)} \longrightarrow T_n^{\pm}\mathbb{F}_p$$

Thm [Miller-Patzt-Putman '21] :
- Always surjective
- Iso for $p = 2, 3, \boxed{5}$.
- not injective for $p \geq 7$.

$$\boxed{\text{Thus} \quad H^{\binom{n}{2}}(\Gamma_n(5)) \cong St_n^{\pm}\mathbb{F}_p}$$

Thm [P.] : Adapt MPP's techniques to compute

$H^{n^2-n}(\Gamma_n(p))$ for :

$p = 3 \in \mathbb{Z}[i]$

$p = 4\omega+1,\ 4\omega+3 \in \mathbb{Z}[\omega]$

These are special instances of a more general result.

Prominently used fact : Eg : $\mathbb{F}_5^{\times}/\{\pm 1\} \cong \mathbb{Z}/2\mathbb{Z}$

$\mathbb{Z}_3[i]^{\times}/\{\pm 1, \pm i\} \cong \mathbb{Z}/2\mathbb{Z}$

Proof idea :
- Generating set for $St_n^{\pm}\mathbb{F}_p \rightsquigarrow$ surjectivity
- Presentation for $St_n^{\pm}\mathbb{F}_p$, $p \leq 5 \rightsquigarrow$ injectivity

using high connectivity of simplicial complexes $BDA_n^{\pm}\mathbb{F}_p$

The general result:

Thm [P.] R Euclidean no. ring, $p \in R$ prime.

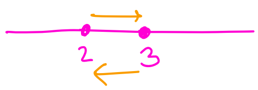
$$\mathbb{F} = R/(p), \quad U = \operatorname{im}(R^\times \longrightarrow \mathbb{F}^\times)$$

$$H^{\nu}(\Gamma_n(p)) \longrightarrow \mathrm{St}_n^U \mathbb{F} \qquad \text{is surj.}$$

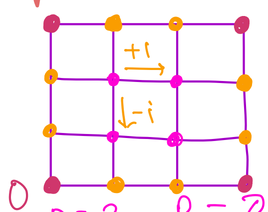
Inj. if $U = \mathbb{F}^\times$, or:
($\text{eg. } p = 3, R = \mathbb{Z}$)

① $\mathbb{F}^\times / U \cong \mathbb{Z}/2$ (eg. $p = 5, R = \mathbb{Z}: (\pm 2)(\pm 2) \equiv \pm 1 \bmod 5$)

② $\mathbb{F}^\times \setminus U$ "additively connected by U"



$p = 5, R = \mathbb{Z}$



$p = 3, R = \mathbb{Z}[i]$

allows us to say different choices of a certain map are homotopic

③ $\mathbb{F}$ additively generated by U allows for an induction argument

④ $2 \in \mathbb{F}^\times \setminus U$ or ________________

⑤ $\mathrm{BDA}_2^U \mathbb{F}$ is 1-connected